Mixing problems in ODEs

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Suppose that we have two chemical substances where one is soluable in the other, such as salt and water. Suppose that we have a tank containing a mixture of these substances, and the mixture of them is poured in and the resulting "well-mixed" solution pours out through a value at the bottom. (The term "well-mixed" is used to indicate that the fluid being poured in is assumed to instantly dissolve into a homogeneous mixture the moment it goes into the tank.) The crude picture looks like this:

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   |     "well-mixed         |
   |         solution"       |
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Assume for concreteness that the chemical substances are salt and water. Let

- $A(t)$ denote the amount of salt at time t,
- "flow-rate-in" = the rate at which the solution pours into the tank,
- "flow-rate-out" = the rate at which the mixture pours out of the tank,
- C_{in} = "concentration in" = the concentration of salt in the solution being poured into the tank,
- C_{out} = "concentration out" = the concentration of salt in the solution being poured out of the tank,

- R_{in} = rate at which the salt is being poured into the tank = ("flow-rate-in")(C_{in}),
- R_{out} = rate at which the salt is being poured out of the tank = ("flow-rate-out")(C_{out}).

Notes: (1) If flow-rate-in = flow-rate-out then the "water level" of the tank stays the same. (2) We can determine C_{out} as a function of other quantities:

$$C_{out} = \frac{A(t)}{T(t)},$$

where $T(t)$ denotes the volume of solution in the tank at time t. (3) The rate of change of the amount of salt in the tank, $A'(t)$, more properly could be called the "net rate of change". If you think if it this way then you see $A'(t) = R_{in} - R_{out}$.

Now the differential equation for the amount of salt arises from the above equations:

$$A'(t) = (\text{``flow} - \text{rate} - \text{in''})C_{in} - (\text{``flow} - \text{rate} - \text{out''})\frac{A(t)}{T(t)}.$$

Example: Consider a tank with 200 liters of salt-water solution, 30 grams of which is salt. Pouring into the tank is a brine solution at a rate of 4 liters/minute and with a concentration of 1 grams per liter. The "well-mixed" solution pours out at a rate of 5 liters/minute. Find the amount at time t.

We know

$$A'(t) = (\text{``flow} - \text{rate} - \text{in''})C_{in} - (\text{``flow} - \text{rate} - \text{out''})\frac{A(t)}{T(t)} = 4 - 5\frac{A(t)}{200 - t}, \quad A(0) = 30.$$

Writing this in the standard form $A' + pA = q$, we have

$$A(t) = \frac{\int \mu(t)q(t)\,dt + C}{\mu(t)},$$

where $\mu = e^{\int p(t)\,dt} = e^{-5\int \frac{1}{200-t}\,dt} = (200 - t)^{-5}$ is the "integrating factor". This gives $A(t) = 200 - t + C \cdot (200 - t)^5$, where the initial condition implies $C = -170 \cdot 200^{-5}$.

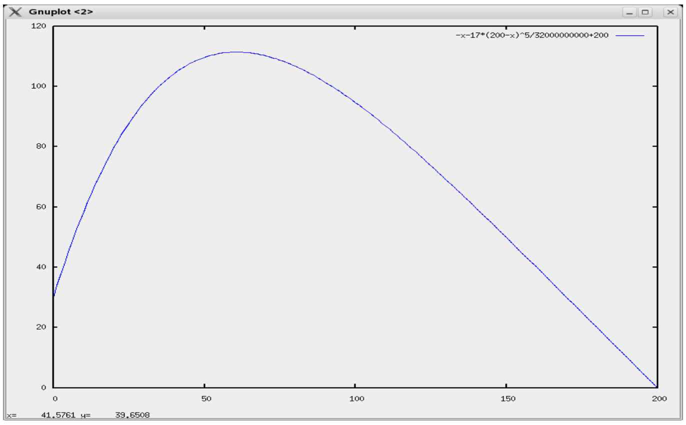

Figure 1: $A(t)$, $0 < t < 50$, $A' = 4 - 5A(t)/(200 - t)$, $A(0) = 30$.

If you now solve the same problem but with the same flow rate out as 4 liters/min (so the "water level" in the tank is constant) then you get $A(t) = 200 - 170e^{-t/50}$, a much different function. This function looks like:

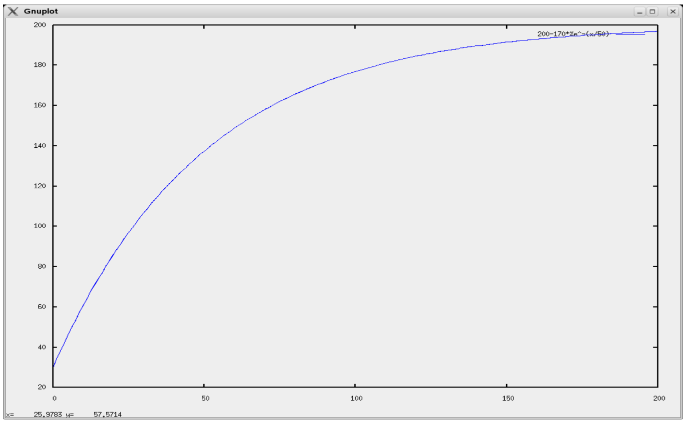

Figure 2: $A(t)$, $0 < t < 50$, $A' = 4 - 4A(t)/200$, $A(0) = 30$.